SOME BERNSTEIN FUNCTIONS AND INTEGRAL REPRESENTATIONS CONCERNING HARMONIC AND GEOMETRIC MEANS

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ABSTRACT. It is general knowledge that the harmonic mean $H(x,y)=\frac{2}{\frac{1}{x}+\frac{1}{y}}$ and that the geometric mean $G(x,y)=\sqrt{xy}$, where x and y are two positive numbers. In the paper, the authors show by several approaches that the harmonic mean $H_{x,y}(t)=H(x+t,y+t)$ and the geometric mean $G_{x,y}(t)=G(x+t,y+t)$ are all Bernstein functions of $t\in(-\min\{x,y\},\infty)$ and establish integral representations of the means $H_{x,y}(t)$ and $G_{x,y}(t)$.

1. Introduction

1.1. **Some definitions.** We recall some notions and definitions.

Definition 1.1 ([17, 27])**.** A function f is said to be completely monotonic on an interval $I\subseteq\mathbb{R}$ if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(t)\ge 0 \tag{1.1}$$

for all $t\in I$ and $n\in\{0\}\cup\mathbb{N}$.

Definition 1.2 ([2])**.** If $f^{(k)}(t)$ for some nonnegative integer k is completely monotonic on an interval $I\subseteq\mathbb{R}$, but $f^{(k-1)}(t)$ is not completely monotonic on I, then $f(t)$ is called a completely monotonic function of k-th order on an interval I.

Definition 1.3 ([20, 22])**.** A function f is said to be logarithmically completely monotonic on an interval $I\subseteq\mathbb{R}$ if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(t)]^{(k)}\ge 0 \tag{1.2}$$

for all $t\in I$ and $k\in\mathbb{N}$.

Definition 1.4 ([25, 27])**.** A function $f:I\subseteq(-\infty,\infty)\to[0,\infty)$ is called a Bernstein function on I if $f(t)$ has derivatives of all orders and $f'(t)$ is completely monotonic on I.

Definition 1.5 ([25])**.** A Stieltjes function is a function $f:(0,\infty)\to[0,\infty)$ which can be written in the form

$$f(x)=\frac{a}{x}+b+\int_0^\infty\frac{1}{s+x}\,\mathrm{d}\,\mu(s), \tag{1.3}$$

where a,b are nonnegative constants and μ is a nonnegative measure on $(0,\infty)$ such that $\int_0^\infty\frac{1}{1+s}\,\mathrm{d}\,\mu(s)<\infty$.

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Definition 1.6 ([9])**.** Let $f(x)$ be a nonnegative function and have derivatives of all orders on $(0,\infty)$. A number $r \in \mathbb{R} \cup \{\pm\infty\}$ is said to be the completely monotonic degree of $f(x)$ with respect to $x \in (0,\infty)$ if $x^r f(x)$ is a completely monotonic function on $(0,\infty)$ but $x^{r+\varepsilon} f(x)$ is not for any positive number $\varepsilon > 0$.

In what follows, for convenience, we denote the sets of completely monotonic functions on $I \subseteq \mathbb{R}$, logarithmically completely monotonic functions on $I \subseteq \mathbb{R}$, Stieltjes functions, and Bernstein functions on $I \subseteq \mathbb{R}$ by $\mathcal{C}[I]$, $\mathcal{L}[I]$, $\mathcal{S}$, and $\mathcal{B}[I]$ respectively.

1.2. **Some relationships and a characterization.** Now we briefly describe some basic relationships between the above defined classes of functions and list a characterization of Bernstein functions on $(0,\infty)$.

In [3, 10, 20, 22], any logarithmically completely monotonic function on an interval I was once again proved to be completely monotonic on I. In [3], the set of all Stieltjes functions was proved to be a subset of all logarithmically completely monotonic functions on $(0,\infty)$. See also [24, Remark 4.8]. Conclusively,

$$\mathcal{S} \subset \mathcal{L}[(0,\infty)] \subset \mathcal{C}[(0,\infty)]. \tag{1.4}$$

It is obvious that any nonnegative completely monotonic function of first order is a Bernstein function.

The relation between Bernstein functions and logarithmically completely monotonic functions was discovered in [7, pp. 161–162, Theorem 3] and [25, p. 45, Proposition 5.17], which reads that the reciprocal of any positive Bernstein function is logarithmically completely monotonic. In other words,

$$0 < f \in \mathcal{B}[I] \Longrightarrow \frac{1}{f} \in \mathcal{L}[I]. \tag{1.5}$$

A relation between $\mathcal{S}$ and $\mathcal{B}[(0,\infty)]$ was given by [4, Theorem 5.4] which may be recited as

$$0 < f \in \mathcal{S} \Longrightarrow \frac{1}{f} \in \mathcal{B}[(0,\infty)]. \tag{1.6}$$

It is easy to see that the degree of any completely monotonic function on $(0,\infty)$ is at least zero. Conversely, if a nonnegative function $f(x)$ on $(0,\infty)$ has a nonnegative degree r, then it must be a completely monotonic function on $(0,\infty)$. See [9, p. 9890].

Bernstein functions can be characterized by [25, p. 15, Theorem 3.2] which states that a function $f : (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if it admits the representation

$$f(x) = a + bx + \int_0^\infty \left(1 - e^{-xt}\right) \mathrm{d}\,\mu(t), \tag{1.7}$$

where $a, b \ge 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_0^\infty \min\{1,t\} \,\mathrm{d}\,\mu(t) < \infty$.

For information on characterizations of the classes $\mathcal{C}[(0,\infty)]$ and $\mathcal{L}[(0,\infty)]$, please refer to related texts in [3, 25, 27] and references cited therein.

1.3. **Some means.** We recall from [26] that the extended mean value $E(r,s;x,y)$ may be defined by

$$E(r,s;x,y) = \left[\frac{r(y^s - x^s)}{s(y^r - x^r)}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y) \ne 0; \tag{1.8}$$

$$
\begin{aligned}
E(r,0;x,y)&=\left[\frac{y^r-x^r}{r(\ln y-\ln x)}\right]^{1/r}, && r(x-y)\ne0; && (1.9)\\
E(r,r;x,y)&=\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, && r(x-y)\ne0; && (1.10)\\
E(0,0;x,y)&=\sqrt{xy}\,, && x\ne y; && (1.11)\\
E(r,s;x,x)&=x, && x=y;
\end{aligned}
$$

where x,y are positive numbers and $r,s\in\mathbb{R}$. Because this mean was first defined in [26], so it is also called Stolarsky's mean by a number of mathematicians. Many special means with two positive variables are special cases of E, for example,

$$
\begin{aligned}
E(r,2r;x,y)&=M_r(x,y), && \text{(power mean)}\\
E(1,p;x,y)&=L_p(x,y), && \text{(generalized logarithmic mean)}\\
E(1,1;x,y)&=I(x,y), && \text{(exponential mean)}\\
E(1,2;x,y)&=A(x,y), && \text{(arithmetic mean)}\\
E(0,0;x,y)&=G(x,y), && \text{(geometric mean)}\\
E(-2,-1;x,y)&=H(x,y), && \text{(harmonic mean)}\\
E(0,1;x,y)&=L(x,y). && \text{(logarithmic mean)}
\end{aligned}
$$

For more information on E, please refer to the monograph [6], the papers [11, 12, 19, 13], and a lot of closely-related references therein.

1.4. **The arithmetic mean is a Bernstein function.** It is easy to see that the arithmetic mean

$$A_{x,y}(t)=A(x+t,y+t)=A(x,y)+t$$

is a trivial Bernstein function of $t\in(-\min\{x,y\},\infty)$ for $x,y>0$.

1.5. **The exponential mean is a Bernstein function.** In [23, p. 116, Remark 6], it was pointed out that,

(1) by standard arguments, it is easy to verify that the reciprocal of the exponential mean

$$I_{x,y}(t)=I(x+t,y+t)=\frac{1}{e}\left[\frac{(x+t)^{x+t}}{(y+t)^{y+t}}\right]^{1/(x-y)} \tag{1.12}$$

for $x,y>0$ with $x\ne y$ is a logarithmically completely monotonic function of $t\in(-\min\{x,y\},\infty)$;

(2) from the newly-discovered integral representation

$$I(x,y)=\exp\left(\frac{1}{y-x}\int_x^y\ln u\,\mathrm{d}\,u\right), \tag{1.13}$$

it is easy to obtain that the exponential mean $I_{x,y}(t)$ for $t>-\min\{x,y\}$ with $x\ne y$ is also a completely monotonic function of first order (that is, a Bernstein function).

1.6. **The logarithmic mean is a Bernstein function.** In [18, p. 616, Remark 3.7], the logarithmic mean

$$L_{x,y}(t)=L(x+t,y+t) \tag{1.14}$$

was proved to be increasing and concave in $t>-\min\{x,y\}$ for $x,y>0$ with $x\ne y$.

More strongly, the logarithmic mean $L_{x,y}(t)$ was proved in [21, Theorem 1] to be a completely monotonic function of first order on $(-\min\{x,y\},\infty)$ for $x,y>0$ with $x\ne y$. Therefore, the logarithmic mean $L_{x,y}(t)$ is a Bernstein function of $t\in(-\min\{x,y\},\infty)$.

Remark 1.1. By [7, pp. 161–162, Theorem 3] or [25, p. 45, Proposition 5.17], the logarithmically complete monotonicity of the exponential mean $I_{x,y}(t)$ and the logarithmic mean $L_{x,y}(t)$ can be deduced respectively from their common property that they are Bernstein functions.

1.7. **Main results.** The goals of this paper are to prove that the harmonic mean

$$H_{x,y}(t)=H(x+t,y+t)=\frac{2}{\frac{1}{x+t}+\frac{1}{y+t}} \tag{1.15}$$

and the geometric mean

$$G_{x,y}(t)=G(x+t,y+t)=\sqrt{(x+t)(y+t)} \tag{1.16}$$

are all Bernstein functions of t on $(-\min\{x,y\},\infty)$ for $x,y>0$ with $x\ne y$, and to establish integral representations of $H_{x,y}(t)$ and $G_{x,y}(t)$.

2. Lemmas

In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *For $i\in\mathbb{N}$, the i-th derivatives of the functions*

$$h(t)=\sqrt{1+\frac{1}{t}}\,, \tag{2.1}$$

the reciprocal $\frac{1}{h(t)}$, and

$$H(t)=h(t)+\frac{1}{h(t)} \tag{2.2}$$

on $(0,\infty)$ may be computed by

$$h^{(i)}(t)=\frac{(-1)^i}{2^it^{i+1}(1+t)^{i-1}h(t)}\sum_{k=0}^{i-1}a_{i,k}t^k, \tag{2.3}$$

$$\left[\frac{1}{h(t)}\right]^{(i)}=\frac{(-1)^{i+1}}{2^it^i(1+t)^ih(t)}\sum_{k=0}^{i-1}b_{i,k}t^k, \tag{2.4}$$

$$H^{(i)}(t)=\frac{(-1)^i}{2^it^{i+1}(1+t)^ih(t)}\sum_{k=0}^{i-1}c_{i,k}t^k, \tag{2.5}$$

where

$$a_{i,k}=\frac{(i-1)!i!(2i-2k-1)!!}{(i-k-1)!(i-k)!k!}2^k, \tag{2.6}$$

$$b_{i,k}=\frac{(i-1)!i!(2i-2k-3)!!}{(i-k-1)!(i-k)!k!}2^k, \tag{2.7}$$

$$c_{i,k}=\frac{(i-1)!(i+1)!(2i-2k-1)!!}{(i-k-1)!(i-k+1)!k!}2^k. \tag{2.8}$$

Consequently, the functions $h(t)$ and $H(t)$ are completely monotonic on $(0,\infty)$, and the reciprocal $\frac{1}{h(t)}$ is a Bernstein function on $(0,\infty)$.

Inductive proof of Lemma 2.1. A direct calculation yields $h'(t)=-\frac{1}{2t^2h(t)}$, which means that

$$a_{1,0}=1. \tag{2.9}$$

So, the formulas (2.3) and (2.6) are valid for $i=1$ and $k=0$.

Differentiating on both sides of (2.3) gives

$$\begin{aligned}h^{(i+1)}(t)=\big[h^{(i)}(t)\big]'&=\left[\frac{(-1)^i}{2^it^{i+1}(1+t)^{i-1}h(t)}\sum_{k=0}^{i-1}a_{i,k}t^k\right]'\\&=\frac{(-1)^{i+1}}{2^{i+1}t^{i+2}(1+t)^ih(t)}\sum_{k=0}^{i-1}[1+2(i-k)+2(2i-k)t]a_{i,k}t^k\\&=\frac{(-1)^{i+1}}{2^{i+1}t^{i+2}(1+t)^ih(t)}\sum_{k=0}^{i}a_{i+1,k}t^k.\end{aligned}$$

Because

$$\begin{aligned}\sum_{k=0}^{i-1}[1+2(i-k)+2(2i-k)t]a_{i,k}t^k&=\sum_{k=0}^{i-1}[1+2(i-k)]a_{i,k}t^k+\sum_{k=0}^{i-1}2(2i-k)a_{i,k}t^{k+1}\\&=\sum_{k=0}^{i-1}[1+2(i-k)]a_{i,k}t^k+\sum_{k=1}^{i}2(2i-k+1)a_{i,k-1}t^k\\&=(1+2i)a_{i,0}+\sum_{k=1}^{i-1}\{[1+2(i-k)]a_{i,k}+2(2i-k+1)a_{i,k-1}\}t^k+2(i+1)a_{i,i-1}t^i,\end{aligned}$$

we obtain

$$a_{i+1,0}=(1+2i)a_{i,0}, \tag{2.10}$$

$$a_{i+1,i}=2ia_{i,i-1}, \tag{2.11}$$

and, for $0<k<i$,

$$a_{i+1,k}=[1+2(i-k)]a_{i,k}+2(2i-k+1)a_{i,k-1}. \tag{2.12}$$

Combining (2.9) with (2.10) and (2.11) results in

$$a_{i,0}=(2i-1)!! \tag{2.13}$$

and

$$a_{i,i-1}=2^{i-1}i!. \tag{2.14}$$

Taking $k=i-1$ in (2.12) and using (2.14) give

$$a_{i+1,i-1}=3a_{i,i-1}+2(i+2)a_{i,i-2}=3\cdot 2^{i-1}i!+2(i+2)a_{i,i-2}. \tag{2.15}$$

From (2.13), it is easily deduced that $a_{2,0}=3$. Substituting this into (2.15) and recurring repeatedly lead to

$$a_{i,i-2}=3(i-1)2^{i-3}i!. \tag{2.16}$$

Taking $k=i-2$ in (2.12) and using (2.16) show

$$a_{i+1,i-2}=5a_{i,i-2}+2(i+3)a_{i,i-3}=15(i-1)2^{i-3}i!+2(i+3)a_{i,i-3}. \tag{2.17}$$

From (2.13), it is readily deduced that $a_{3,0}=15$. Substituting this into (2.17) and recurring repeatedly reveal

$$a_{i,i-3}=5(i-2)(i-1)2^{i-5}i!. \tag{2.18}$$

Taking $k=i-3$ in (2.12) and using (2.18) show

$$a_{i+1,i-3}=7a_{i,i-3}+2(i+4)a_{i,i-4}=35(i-2)(i-1)2^{i-5}i!+2(i+4)a_{i,i-4}. \tag{2.19}$$

From (2.13), it is immediately obtained that $a_{4,0}=105$. Substituting this into (2.19) and recurring repeatedly yield

$$a_{i,i-4}=\frac{35}{3}(i-3)(i-2)(i-1)2^{i-8}i!. \tag{2.20}$$

By the same arguments as above, we may obtain

$$a_{i,i-5}=21(i-4)(i-3)(i-2)(i-1)2^{i-11}i! \tag{2.21}$$

and

$$a_{i,i-6}=\frac{77}{5}(i-5)(i-4)(i-3)(i-2)(i-1)2^{i-13}i!. \tag{2.22}$$

Inductively, we can derive that

$$a_{i,i-k}=\lambda_{i,i-k}\frac{(i-1)!}{(i-k)!}2^{i-k}i! \tag{2.23}$$

for $0<k<i$. Specially, we have

$$\begin{aligned}
&\lambda_{i,i-1}=1, && \lambda_{i,i-2}=\frac{3}{2}, && \lambda_{i,i-3}=\frac{5}{4},\\
&\lambda_{i,i-4}=\frac{35}{3\cdot 2^4}, && \lambda_{i,i-5}=\frac{21}{2^6}, && \lambda_{i,i-6}=\frac{77}{5\cdot 2^7}.
\end{aligned} \tag{2.24}$$

Replacing k by $i-\ell$ in (2.23) yields

$$a_{i,\ell}=\lambda_{i,\ell}\frac{(i-1)!}{\ell!}2^{\ell}i! \tag{2.25}$$

for $0<\ell<i$. Substituting (2.25) into (2.12) leads to

$$[1+2(i-\ell)]\lambda_{i,\ell}+\ell(2i-\ell+1)\lambda_{i,\ell-1}=i(i+1)\lambda_{i+1,\ell} \tag{2.26}$$

for $0<\ell<i$. The equality (2.26) is equivalent to

$$(1+2k)\lambda_{i,i-k}+(i-k)(i+k+1)\lambda_{i,i-k-1}=i(i+1)\lambda_{i+1,i-k} \tag{2.27}$$

for $0<k<i$.

The quantities in (2.24) implies that $\lambda_{i,i-k}=\mu_k$, that is, $\lambda_{i,i-k}$ is independent of i. Then the equality (2.27) may be written as

$$(1+2k)\mu_k=[i(i+1)-(i-k)(i+k+1)]\mu_{k+1}=k(1+k)\mu_{k+1} \tag{2.28}$$

for $0<k<i$. Recurring (2.28) by $\mu_1=\lambda_{i,i-1}=1$ reveals

$$\mu_k=\lambda_{i,i-k}=\frac{(2k-1)!!}{(k-1)!k!} \tag{2.29}$$

for $0<k<i$. As a result, by (2.29), we conclude that

$$a_{i,i-k}=\frac{(2k-1)!!}{(k-1)!k!}\frac{(i-1)!}{(i-k)!}2^{i-k}i! \tag{2.30}$$

for $0<k<i$. Replacing k by $i-\ell$ in (2.30) shows

$$a_{i,\ell}=\frac{(2i-2\ell-1)!!}{(i-\ell-1)!(i-\ell)!}\frac{(i-1)!}{\ell!}2^\ell i! \tag{2.31}$$

for $0<\ell<i$. It is easy to verify that the sequence (2.31) for $0\le \ell\le i-1$ meets the recursion formulas (2.10), (2.11), and (2.12). The formulas (2.3) and (2.6) for general terms are thus proved.

It is obvious that $h'(t)=-\frac{1}{2t^2h(t)}$ which is equivalent to $\frac{1}{h(t)}=-2t^2h'(t)$. Therefore, using the formulas (2.3) and (2.6) just verified, we have

$$\begin{aligned}
\left[\frac{1}{h(t)}\right]^{(i)}&=-2\bigl[t^2h'(t)\bigr]^{(i)}\\
&=-2\sum_{\ell=0}^{i}\binom{i}{\ell}\bigl(t^2\bigr)^{(\ell)}h^{(i-\ell+1)}(t)\\
&=-2\left[\binom{i}{0}t^2h^{(i+1)}(t)+2\binom{i}{1}th^{(i)}(t)+2\binom{i}{2}h^{(i-1)}(t)\right]\\
&=-2\left[\frac{(-1)^{i+1}}{2^{i+1}t^i(1+t)^ih(t)}\sum_{k=0}^{i}a_{i+1,k}t^k+\frac{(-1)^i i}{2^{i-1}t^i(1+t)^{i-1}h(t)}\sum_{k=0}^{i-1}a_{i,k}t^k\right.\\
&\quad\left.+\frac{(-1)^{i-1}(i-1)i}{2^{i-1}t^i(1+t)^{i-2}h(t)}\sum_{k=0}^{i-2}a_{i-1,k}t^k\right]\\
&=\frac{(-1)^{i+1}}{2^it^i(1+t)^ih(t)}\left[4i(1+t)\sum_{k=0}^{i-1}a_{i,k}t^k-\sum_{k=0}^{i}a_{i+1,k}t^k\right.\\
&\quad\left.-4(i-1)i(1+t)^2\sum_{k=0}^{i-2}a_{i-1,k}t^k\right]\\
&=\frac{(-1)^{i+1}}{2^it^i(1+t)^ih(t)}\Bigl\{4ia_{i,0}-4i(i-1)a_{i-1,0}-a_{i+1,0}\\
&\quad+[4i(a_{i,1}+a_{i,0})-4i(i-1)(a_{i-1,1}+2a_{i-1,0})-a_{i+1,1}]t\\
&\quad+[4i(a_{i,i-1}+a_{i,i-2})-4i(i-1)(a_{i-1,i-3}+2a_{i-1,i-2})-a_{i+1,i-1}]t^{i-1}\\
&\quad+[4ia_{i,i-1}-4i(i-1)a_{i-1,i-2}-a_{i+1,i}]t^i+\sum_{k=2}^{i-2}[4i(a_{i,k}+a_{i,k-1})\\
&\quad-4i(i-1)(a_{i-1,k}+2a_{i-1,k-1}+a_{i-1,k-2})-a_{i+1,k}]t^k\Bigr\}\\
&=\frac{(-1)^{i+1}}{2^it^i(1+t)^ih(t)}\sum_{k=0}^{i-1}\frac{(i-1)!i!(2i-2k-3)!!}{(i-k-1)!(i-k)!k!}2^kt^k.
\end{aligned}$$

Hence, the general formulas (2.4) and (2.7) are obtained.

Adding the two formulas (2.3) and (2.4) yields

$$h^{(i)}(t)+\left[\frac{1}{h(t)}\right]^{(i)}=\frac{(-1)^i}{2^it^{i+1}(1+t)^ih(t)}\left[(1+t)\sum_{k=0}^{i-1}a_{i,k}t^k-t\sum_{k=0}^{i-1}b_{i,k}t^k\right]$$

$$
\begin{aligned}
&= \frac{(-1)^i}{2^i t^{i+1}(1+t)^i h(t)}\left[\sum_{k=0}^{i-1}(a_{i,k}-b_{i,k})t^{k+1}+\sum_{k=0}^{i-1}a_{i,k}t^k\right]\\
&= \frac{(-1)^i}{2^i t^{i+1}(1+t)^i h(t)}\left[\sum_{k=1}^{i}(a_{i,k-1}-b_{i,k-1})t^{k}+\sum_{k=0}^{i-1}a_{i,k}t^k\right]\\
&= \frac{(-1)^i}{2^i t^{i+1}(1+t)^i h(t)}\left[a_{i,0}+\sum_{k=1}^{i-1}(a_{i,k-1}-b_{i,k-1}+a_{i,k})t^{k}+(a_{i,i-1}-b_{i,i-1})t^i\right]\\
&= \frac{(-1)^i}{2^i t^{i+1}(1+t)^i h(t)}\left\{(2i-1)!!+\sum_{k=1}^{i-1}\frac{(i-1)!(i+1)!(2i-2k-1)!!}{(i-k-1)!(i-k+1)!k!}2^k t^k\right\}\\
&= \frac{(-1)^i}{2^i t^{i+1}(1+t)^i h(t)}\sum_{k=0}^{i-1}\frac{(i-1)!(i+1)!(2i-2k-1)!!}{(i-k-1)!(i-k+1)!k!}2^k t^k.
\end{aligned}
$$

This implies that the function $H(t)$ is completely monotonic on $(0,\infty)$. The proof of Lemma 2.1 is completed. $\square$

Short proofs of a part of Lemma 2.1. In [25, p. 13, Remark 2.4], it was collected as an example that the function $\frac{1}{a+t}$ is a Stieltjes function for $a>0$. The property (iv) in Section 3 of [4] (See also the property (vii) in [16, Theorem 1.3]) reads that if $f\in\mathcal{S}$ then $f^\alpha\in\mathcal{S}$ for $0\le\alpha\le1$. Specially for $a=1$ and $\alpha=\frac12$, we have $h_1(t)=\frac{1}{\sqrt{1+t}}\in\mathcal{S}$. The property (i) in Section 3 of [4] (See also the property (i) in [16, Theorem 1.3]) states that if $f\in\mathcal{S}\setminus\{0\}$ then $\frac{1}{f(1/x)}\in\mathcal{S}$. Applying this property to $h_1(t)$ brings out

$$
h(t)=\frac{1}{h_1(1/t)}\in\mathcal{S} \tag{2.32}
$$

which means, by the relation from the very ends of the inclusions (1.4), that $h(t)\in\mathcal{C}[(0,\infty)]$ and, by the relation (1.6), that $\frac{1}{h(t)}\in\mathcal{B}[(0,\infty)]$.

In [25, p. 24, Remark 3.11], it was listed as examples that $h_2(t)=t^\beta\in\mathcal{B}[(0,\infty)]$ for $0<\beta<1$ and $h_3(t)=\frac{t}{1+t}\in\mathcal{B}[(0,\infty)]$. The item (iii) of Corollary 3.7 in [25, p. 20] write that if $f_1,f_2\in\mathcal{B}[(0,\infty)]$ then $f_1\circ f_2\in\mathcal{B}[(0,\infty)]$. Applying f_1 and f_2 respectively to h_2 and h_3 reveals once again that $\frac{1}{h(t)}=\sqrt{\frac{t}{1+t}}\in\mathcal{B}[(0,\infty)]$.

Taking $h_3(x)=x+\frac1x$ and $h_4(t)=\frac{1}{h(t)}=\frac{1}{\sqrt{1+1/t}}$. It is easy to see that $h_3\in\mathcal{C}[(0,1)]$ and $0<h_4(t)<1$. A part of Theorem 3.6 in [25, p. 19] asserts that if $0<f\in\mathcal{B}[(0,\infty)]$ then $g\circ f\in\mathcal{C}[(0,\infty)]$ for every $g\in\mathcal{C}[(0,\infty)]$. Since $h_4\in\mathcal{B}[(0,\infty)]$, applying f and g in this assertion respectively to h_4 and h_3 leads to $H(t)=h(t)+\frac{1}{h(t)}\in\mathcal{C}[(0,\infty)]$. The proof of Lemma 2.1 is completed. $\square$

Lemma 2.2. *For $z\in\mathbb{C}\setminus(-\infty,0]$, the complex functions $h(z)$ and $\frac{1}{h(z)}$ have integral representations*

$$
h(z)=1+\frac1\pi\int_0^1\sqrt{\frac1u-1}\,\frac{\mathrm{d}\,u}{u+z} \tag{2.33}
$$

and

$$
\frac{1}{h(z)}=1-\frac1\pi\int_0^1\frac{1}{\sqrt{\frac1u-1}}\,\frac{\mathrm{d}\,u}{u+z}. \tag{2.34}
$$

Consequently, the functions $h(t)$ and $1-\frac{1}{h(t)}$ are Stieltjes functions and the complex function $H(z)$ has the integral integral representation

$$H(z)=2+\frac{1}{\pi}\int_0^\infty \rho(s)e^{-zs}\,\mathrm{d}\,s \tag{2.35}$$

for $z\in\mathbb{C}\setminus(-\infty,0]$, where

$$\rho(s)=\int_0^{1/2}q(u)\bigl[1-e^{-(1-2u)s}\bigr]e^{-us}\,\mathrm{d}\,u=\int_0^{1/2}q\biggl(\frac12-u\biggr)\bigl(e^{us}-e^{-us}\bigr)e^{-s/2}\,\mathrm{d}\,u \tag{2.36}$$

is nonnegative on $(0,\infty)$ and

$$q(u)=\sqrt{\frac1u-1}-\frac{1}{\sqrt{1/u-1}} \tag{2.37}$$

on $(0,1)$.

Proof by Cauchy integral formula. By standard arguments, we immediately obtain that

$$\lim_{z\to0}[zh(z)]=\lim_{z\to0}\sqrt{z^2+z}=\sqrt{\lim_{z\to0}(z^2+z)}=0, \tag{2.38}$$

$$\lim_{z\to0}\frac{z}{h(z)}=\lim_{z\to0}\sqrt{\frac{z^3}{1+z}}=\sqrt{\lim_{z\to0}\frac{z^3}{1+z}}=0, \tag{2.39}$$

$$\lim_{z\to\infty}\sqrt{1+\frac1z}=\sqrt{1+\lim_{z\to\infty}\frac1z}=1, \tag{2.40}$$

$$\lim_{z\to\infty}\frac{1}{\sqrt{1+\frac1z}}=\frac{1}{\sqrt{1+\lim_{z\to\infty}\frac1z}}=1, \tag{2.41}$$

$$h(\overline{z})=\overline{h(z)}, \tag{2.42}$$

$$\frac{1}{h(\overline{z})}=\overline{\biggl[\frac{1}{h(z)}\biggr]}. \tag{2.43}$$

For $t\in(0,\infty)$ and $\varepsilon>0$, we have

$$\begin{aligned}
h(-t+i\varepsilon)&=\sqrt{1+\frac{1}{-t+i\varepsilon}}=\sqrt{1+\frac{-t-i\varepsilon}{t^2+\varepsilon^2}}=\exp\biggl[\frac12\ln\biggl(1+\frac{-t-i\varepsilon}{t^2+\varepsilon^2}\biggr)\biggr]\\
&=\exp\biggl\{\frac12\biggl[\ln\biggl|\frac{t^2+\varepsilon^2-t}{t^2+\varepsilon^2}-i\frac{\varepsilon}{t^2+\varepsilon^2}\biggr|+i\arg\biggl(\frac{t^2+\varepsilon^2-t}{t^2+\varepsilon^2}-i\frac{\varepsilon}{t^2+\varepsilon^2}\biggr)\biggr]\biggr\}\\
&=\exp\biggl\{\frac12\biggl[\ln p(t,\varepsilon)+i\arg\biggl(\frac{t^2+\varepsilon^2-t}{t^2+\varepsilon^2}-i\frac{\varepsilon}{t^2+\varepsilon^2}\biggr)\biggr]\biggr\}\\
&=\begin{cases}
\exp\biggl\{\dfrac12\biggl[\ln p(t,\varepsilon)+i\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}\biggr]\biggr\}, & t^2+\varepsilon^2-t>0,\\
\exp\biggl\{\dfrac12\biggl[\ln p(t,\varepsilon)+i\biggl(\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}-\pi\biggr)\biggr]\biggr\}, & t^2+\varepsilon^2-t<0,\\
\exp\biggl\{\dfrac12\biggl(\ln\dfrac{\varepsilon}{t^2+\varepsilon^2}-i\dfrac{\pi}{2}\biggr)\biggr\}, & t^2+\varepsilon^2-t=0,
\end{cases}
\end{aligned}$$

where

$$p(t,\varepsilon)=\sqrt{\biggl(\frac{t^2+\varepsilon^2-t}{t^2+\varepsilon^2}\biggr)^2+\biggl(\frac{\varepsilon}{t^2+\varepsilon^2}\biggr)^2}\,.$$

Hence,

$$\Im h(-t+i\varepsilon)=\begin{cases}\exp\left[\dfrac12\ln p(t,\varepsilon)\right]\sin\left(\dfrac12\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}\right), & t^2+\varepsilon^2-t>0;\\ \exp\left[\dfrac12\ln p(t,\varepsilon)\right]\sin\left(\dfrac12\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}-\dfrac{\pi}{2}\right), & t^2+\varepsilon^2-t<0;\\ -\exp\left(\dfrac12\ln\dfrac{\varepsilon}{t^2+\varepsilon^2}\right)\sin\dfrac{\pi}{4}, & t^2+\varepsilon^2-t=0.\end{cases}$$

Accordingly,

$$\lim_{\varepsilon\to0^+}\Im h(-t+i\varepsilon)=\begin{cases}-\sqrt{\dfrac1t-1}, & 0<t<1;\\ \infty, & t=1;\\ 0, & t>1.\end{cases}\tag{2.44}$$

Similarly, for $t\in(0,\infty)$ and $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{h(-t+i\varepsilon)}&=\exp\left[-\frac12\ln\left(1+\frac{-t-i\varepsilon}{t^2+\varepsilon^2}\right)\right]\\&=\begin{cases}\exp\left\{-\dfrac12\left[\ln p(t,\varepsilon)+i\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}\right]\right\}, & t^2+\varepsilon^2-t>0;\\ \exp\left\{-\dfrac12\left[\ln p(t,\varepsilon)+i\left(\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}-\pi\right)\right]\right\}, & t^2+\varepsilon^2-t<0;\\ \exp\left\{-\dfrac12\left(\ln\dfrac{\varepsilon}{t^2+\varepsilon^2}-i\dfrac{\pi}{2}\right)\right\}, & t^2+\varepsilon^2-t=0.\end{cases}\end{aligned}$$

Therefore,

$$\Im\left[\frac{1}{h(-t+i\varepsilon)}\right]=\begin{cases}-\exp\left[-\dfrac12\ln p(t,\varepsilon)\right]\sin\left(\dfrac12\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}\right), & t^2+\varepsilon^2-t>0;\\ -\exp\left[-\dfrac12\ln p(t,\varepsilon)\right]\sin\left(\dfrac12\arctan\dfrac{\varepsilon}{t^2+\varepsilon^2}-\dfrac{\pi}{2}\right), & t^2+\varepsilon^2-t<0;\\ \exp\left(-\dfrac12\ln\dfrac{\varepsilon}{t^2+\varepsilon^2}\right)\sin\dfrac{\pi}{4}, & t^2+\varepsilon^2-t=0.\end{cases}$$

Consequently,

$$\lim_{\varepsilon\to0^+}\Im\left[\frac{1}{h(-t+i\varepsilon)}\right]=\begin{cases}\sqrt{\dfrac{t}{1-t}}, & 0<t<1;\\ \infty, & t=1;\\ 0, & t>1.\end{cases}\tag{2.45}$$

Let D be a bounded domain with piecewise smooth boundary. The famous Cauchy integral formula (See [8, p. 113]) reads that if $f(z)$ is analytic on D, and $f(z)$ extends smoothly to the boundary of D, then

$$f(z)=\frac{1}{2\pi i}\oint_{\partial D}\frac{f(w)}{w-z}\,\mathrm{d}\,w,\quad z\in D.\tag{2.46}$$

For any fixed point $z\in\mathbb{C}\setminus(-\infty,0]$, choose $0<\varepsilon<1$ and $r>0$ such that $0<\varepsilon<|z|<r$, and consider the positively oriented contour $C(\varepsilon,r)$ in $\mathbb{C}\setminus(-\infty,0]$ consisting of the half circle $z=\varepsilon e^{i\theta}$ for $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and the half lines $z=x\pm i\varepsilon$

for $x \le 0$ until they cut the circle $|z| = r$, which close the contour at the points $-r(\varepsilon) \pm i\varepsilon$, where $0 < r(\varepsilon) \to r$ as $\varepsilon \to 0$. See Figure 1.

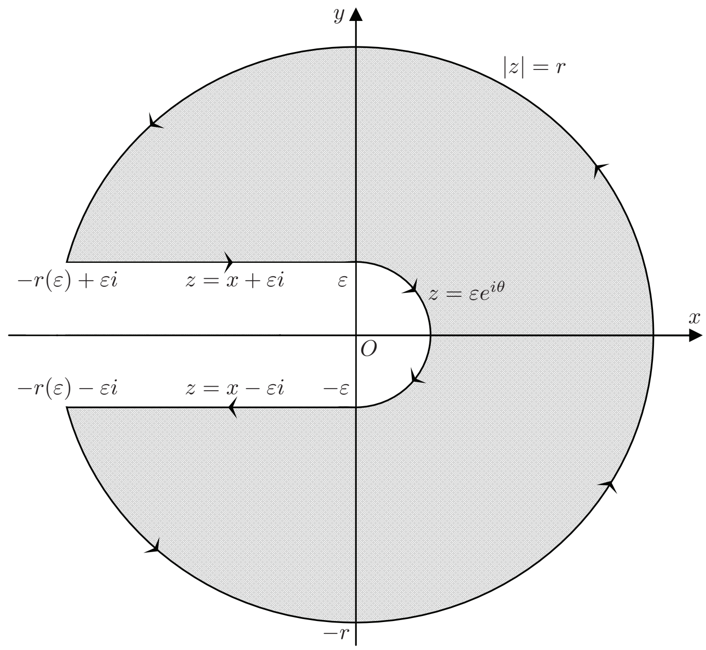

FIGURE 1. The contour $C(\varepsilon, r)$

By the above mentioned Cauchy integral formula, we have

$$
\begin{aligned}
h(z) &= \frac{1}{2\pi i} \oint_{C(\varepsilon,r)} \frac{h(w)}{w-z} \,\mathrm{d}w \\
&= \frac{1}{2\pi i} \biggl[\int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta} h\bigl(\varepsilon e^{i\theta}\bigr)}{\varepsilon e^{i\theta} - z} \,\mathrm{d}\theta + \int_{-r(\varepsilon)}^{0} \frac{h(x+i\varepsilon)}{x+i\varepsilon-z} \,\mathrm{d}x \\
&\quad + \int_{0}^{-r(\varepsilon)} \frac{h(x-i\varepsilon)}{x-i\varepsilon-z} \,\mathrm{d}x + \int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]} \frac{ire^{i\theta} h\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z} \,\mathrm{d}\theta \biggr].
\end{aligned} \tag{2.47}
$$

By the limit (2.38), it follows that

$$
\lim_{\varepsilon \to 0^+} \int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta} h\bigl(\varepsilon e^{i\theta}\bigr)}{\varepsilon e^{i\theta} - z} \,\mathrm{d}\theta = 0. \tag{2.48}
$$

In virtue of the limit (2.40), it can be derived that

$$
\lim_{\substack{\varepsilon \to 0^+ \\ r \to \infty}} \int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]} \frac{ire^{i\theta} h\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z} \,\mathrm{d}\theta = \lim_{r \to \infty} \int_{-\pi}^{\pi} \frac{ire^{i\theta} h\bigl(re^{i\theta}\bigr)}{re^{i\theta}-z} \,\mathrm{d}\theta = 2\pi i. \tag{2.49}
$$

Making use of the limits (2.42) and (2.44) yields that

$$
\int_{-r(\varepsilon)}^{0} \frac{h(x+i\varepsilon)}{x+i\varepsilon-z} \,\mathrm{d}x + \int_{0}^{-r(\varepsilon)} \frac{h(x-i\varepsilon)}{x-i\varepsilon-z} \,\mathrm{d}x = \int_{-r(\varepsilon)}^{0} \biggl[\frac{h(x+i\varepsilon)}{x+i\varepsilon-z} - \frac{h(x-i\varepsilon)}{x-i\varepsilon-z} \biggr] \,\mathrm{d}x
$$

$$
\begin{aligned}
&= \int_{-r(\varepsilon)}^{0} \frac{(x-i\varepsilon-z)h(x+i\varepsilon)-(x+i\varepsilon-z)h(x-i\varepsilon)}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x \\
&= \int_{-r(\varepsilon)}^{0} \frac{(x-z)[h(x+i\varepsilon)-h(x-i\varepsilon)]-i\varepsilon[h(x-i\varepsilon)+h(x+i\varepsilon)]}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x \\
&= 2i\int_{-r(\varepsilon)}^{0} \frac{(x-z)\Im h(x+i\varepsilon)-\varepsilon\Re h(x+i\varepsilon)}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x \\
&\to 2i\int_{-r}^{0} \frac{\lim_{\varepsilon\to0^+}\Im h(x+i\varepsilon)}{x-z}\,\mathrm{d}x \\
&= -2i\int_{0}^{r} \frac{\lim_{\varepsilon\to0^+}\Im h(-t+i\varepsilon)}{t+z}\,\mathrm{d}t \\
&\to -2i\int_{0}^{\infty} \frac{\lim_{\varepsilon\to0^+}\Im h(-t+i\varepsilon)}{t+z}\,\mathrm{d}t \\
&= 2i\int_{0}^{1}\sqrt{\frac{1}{t}-1}\,\frac{\mathrm{d}t}{t+z}
\end{aligned}
\tag{2.50}
$$

as $\varepsilon\to0^+$ and $r\to\infty$. Substituting equations (2.48), (2.49), and (2.50) into (2.47) and simplifying produce the integral representation (2.33).

Similarly, by the above mentioned Cauchy integral formula, we have

$$
\begin{aligned}
\frac{1}{h(z)} &= \frac{1}{2\pi i}\oint_{C(\varepsilon,r)} \frac{1/h(w)}{w-z}\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\biggl[\int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta}\bigl[1/h\bigl(\varepsilon e^{i\theta}\bigr)\bigr]}{\varepsilon e^{i\theta}-z}\,\mathrm{d}\theta + \int_{-r(\varepsilon)}^{0} \frac{1/h(x+i\varepsilon)}{x+i\varepsilon-z}\,\mathrm{d}x \\
&\quad + \int_{0}^{-r(\varepsilon)} \frac{1/h(x-i\varepsilon)}{x-i\varepsilon-z}\,\mathrm{d}x + \int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]} \frac{ire^{i\theta}\bigl[1/h\bigl(re^{i\theta}\bigr)\bigr]}{re^{i\theta}-z}\,\mathrm{d}\theta\biggr].
\end{aligned}
\tag{2.51}
$$

From the limit (2.39), it follows that

$$
\lim_{\varepsilon\to0^+}\int_{\pi/2}^{-\pi/2} \frac{i\varepsilon e^{i\theta}\bigl[1/h\bigl(\varepsilon e^{i\theta}\bigr)\bigr]}{\varepsilon e^{i\theta}-z}\,\mathrm{d}\theta = 0. \tag{2.52}
$$

By virtue of the limit (2.41), it may be deduced that

$$
\lim_{\substack{\varepsilon\to0^+\\ r\to\infty}}\int_{\arg[-r(\varepsilon)-i\varepsilon]}^{\arg[-r(\varepsilon)+i\varepsilon]} \frac{ire^{i\theta}\bigl[1/h\bigl(re^{i\theta}\bigr)\bigr]}{re^{i\theta}-z}\,\mathrm{d}\theta = 2\pi i. \tag{2.53}
$$

Employing the limits (2.43) and (2.45) yields that

$$
\begin{aligned}
&\int_{-r(\varepsilon)}^{0} \frac{1/h(x+i\varepsilon)}{x+i\varepsilon-z}\,\mathrm{d}x + \int_{0}^{-r(\varepsilon)} \frac{1/h(x-i\varepsilon)}{x-i\varepsilon-z}\,\mathrm{d}x \\
&= 2i\int_{-r(\varepsilon)}^{0} \frac{(x-z)\Im[1/h(x+i\varepsilon)]-\varepsilon\Re[1/h(x+i\varepsilon)]}{(x+i\varepsilon-z)(x-i\varepsilon-z)}\,\mathrm{d}x \\
&\to 2i\int_{-r}^{0} \frac{\lim_{\varepsilon\to0^+}\Im[1/h(x+i\varepsilon)]}{x-z}\,\mathrm{d}x \quad \text{as } \varepsilon\to0^+ \\
&\to -2i\int_{0}^{\infty} \frac{\lim_{\varepsilon\to0^+}\Im h(-t+i\varepsilon)}{t+z}\,\mathrm{d}t \quad \text{as } r\to\infty
\end{aligned}
$$

$$= -2i\int_0^1 \sqrt{\frac{t}{1-t}}\,\frac{\mathrm{d}\,t}{t+z}. \tag{2.54}$$

Substituting equations (2.52), (2.53), and (2.54) into (2.51) and simplifying produce the integral representation (2.34).

Adding (2.33) and (2.34) leads to

$$\begin{aligned} H(z) &= 2+\frac{1}{\pi}\int_0^1 q(u)\frac{\mathrm{d}\,u}{u+z} \\ &= 2+\frac{1}{\pi}\int_0^1 q(u)\int_0^\infty e^{-(u+z)s}\,\mathrm{d}\,s\,\mathrm{d}\,u \\ &= 2+\frac{1}{\pi}\int_0^\infty\biggl[\int_0^1 q(u)e^{-us}\,\mathrm{d}\,u\biggr]e^{-zs}\,\mathrm{d}\,s. \end{aligned}$$

Utilizing $q(u)=-q(1-u)$ for $u\in(0,1)$ or $q\bigl(\frac12+u\bigr)=-q\bigl(\frac12-u\bigr)$ for $u\in\bigl(0,\frac12\bigr)$ results in

$$\begin{aligned} \int_0^1 q(u)e^{-us}\,\mathrm{d}\,u &= \int_0^{1/2} q(u)e^{-us}\,\mathrm{d}\,u+\int_{1/2}^1 q(u)e^{-us}\,\mathrm{d}\,u \\ &= \int_0^{1/2} q(u)e^{-us}\,\mathrm{d}\,u+\int_0^{1/2} q(1-u)e^{-(1-u)s}\,\mathrm{d}\,u \\ &= \int_0^{1/2} q(u)\bigl[e^{-us}-e^{-(1-u)s}\bigr]\,\mathrm{d}\,u = \int_0^{1/2} q(u)\bigl[1-e^{-(1-2u)s}\bigr]e^{-us}\,\mathrm{d}\,u \ge 0 \end{aligned}$$

or

$$\begin{aligned} \int_0^1 q(u)e^{-us}\,\mathrm{d}\,u &= \int_0^{1/2} q\biggl(\frac12-u\biggr)e^{-(1/2-u)s}\,\mathrm{d}\,u+\int_0^{1/2} q\biggl(\frac12+u\biggr)e^{-(1/2+u)s}\,\mathrm{d}\,u \\ &= \int_0^{1/2} q\biggl(\frac12-u\biggr)\bigl[e^{-(1/2-u)s}-e^{-(1/2+u)s}\bigr]\,\mathrm{d}\,u \\ &= \int_0^{1/2} q\biggl(\frac12-u\biggr)\bigl(e^{us}-e^{-us}\bigr)e^{-s/2}\,\mathrm{d}\,u \\ &\ge 0. \end{aligned}$$

The proof of Lemma 2.2 is thus completed. □

Proof by Stieltjes-Perron inversion formula. The property (x) in [16, Theorem 1.3] formulates that if $f\in\mathcal{S}$ then $f^\alpha(0^+)-f^\alpha\bigl(\frac1t\bigr)\in\mathcal{S}$ for $0\le\alpha\le1$. Since $h(t)\in\mathcal{S}$, see (2.32), and, by the property (i) in [16, Theorem 1.3], $\frac{1}{h(1/t)}\in\mathcal{S}$, replacing f by $\frac{1}{h(1/t)}$, making use of the easy fact that $f(0^+)=\lim_{t\to0^+}f(t)=1$, and letting $\alpha=1$ yield $1-\frac{1}{h(t)}\in\mathcal{S}$.

For a Stieltjes function f given by (1.3), by the Stieltjes-Perron inversion formula in [14, p. 591], we can determine the scalars $a=\lim_{x\to0^+}[xf(x)]$ and $b=\lim_{x\to\infty}f(x)$ and the measure

$$\mu(s) = -\frac{1}{\pi}\lim_{t\to0^+}\Im\int_{-\infty}^{-s} f(u+ti)\,\mathrm{d}\,u, \tag{2.55}$$

as done in [3, 15]. Specially, for the function $h(x)$, since $a=\lim_{x\to0^+}[xh(x)]=0$ and $b=\lim_{x\to\infty}h(x)=1$, we have

$$h(z)=1+\int_0^\infty\frac{\mathrm{d}\,\Phi(u)}{u+z}\tag{2.56}$$

for $|\arg z|<\pi$, where

$$\Phi(u)=\frac1\pi\lim_{s\to0^+}\int_u^\infty\Im\sqrt{1-\frac{1}{\tau-is}}\,\mathrm{d}\,\tau=-\frac1\pi\int_u^\infty\sqrt{\frac1\tau-1}\,\mathrm{d}\,\tau$$

when $0<\tau<1$ and $\Phi(u)=0$ when $\tau>1$ because taking $s\to0^+$ we obtain

$$\Im\sqrt{1-\frac{1}{\tau-is}}=\Im\sqrt{\frac{-[\tau(1-\tau)-s^2]-si}{\tau^2+s^2}}\to-\sqrt{\frac{\tau(1-\tau)}{\tau^2}}$$

when $0<\tau<1$ and $\Im\sqrt{1-\frac{1}{\tau-si}}\to0$ when $\tau>1$. Thus we find

$$\Phi'(u)=\frac1\pi\sqrt{\frac1u-1}$$

when $0<u<1$ and $\Phi'(u)=0$ when $u>1$. Substituting $\Phi'(u)$ in the representation (2.56) results in the formula (2.33).

The formula (2.34) for $\frac{1}{h(z)}$ or for $1-\frac{1}{h(z)}$ can be derived in a similar way as above.

The rest is the same as in the first proof. Lemma 2.2 is proved once again. □

3. The harmonic mean is a Bernstein function

Our results on the harmonic mean $H_{x,y}(t)$ may be stated as the theorem below.

Theorem 3.1. *The harmonic mean $H_{x,y}(t)$ defined by (1.15) is a Bernstein function of t on $(-\min\{x,y\},\infty)$ for $x,y>0$ with $x\ne y$ and has the integral representation*

$$H_{x,y}(t)=H(x,y)+t+\frac{(x-y)^2}{4}\int_0^\infty\bigl(1-e^{-tu}\bigr)e^{-(x+y)u/2}\,\mathrm{d}\,u.\tag{3.1}$$

Consequently,

$$H(x,y)=A(x,y)-\frac{(x-y)^2}{2}\int_0^\infty e^{-(x+y)u}\,\mathrm{d}\,u\tag{3.2}$$

$$H(s,y+s)=s+\frac{y^2}{4}\int_0^\infty\bigl(1-e^{-su}\bigr)e^{-yu/2}\,\mathrm{d}\,u,\quad s>0.\tag{3.3}$$

Proof. The harmonic mean $H_{x,y}(t)$ meets

$$H'_{x,y}(t)=\frac{2\bigl[x^2+y^2+2(x+y)t+2t^2\bigr]}{(x+y+2t)^2}=1+\frac{(x-y)^2}{(x+y+2t)^2}>1.\tag{3.4}$$

It is obvious that the derivative $H'_{x,y}(t)$ is completely monotonic with respect to t. As a result, the harmonic mean $H_{x,y}(t)$ is a Bernstein function of t on $(-\min\{x,y\},\infty)$ for $x,y>0$ with $x\ne y$.

In [1, p. 255, 6.1.1], it was listed that, for $\Re z>0$ and $\Re k>0$, the classical Euler gamma function

$$\Gamma(z)=k^z\int_0^\infty t^{z-1}e^{-kt}\,\mathrm{d}\,t.\tag{3.5}$$

This formula can be rearranged as

$$\frac{1}{z^w} = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1} e^{-zt} \,\mathrm{d}\, t \tag{3.6}$$

for $\Re z > 0$ and $\Re w > 0$. Combining (3.6) with (3.4) yields

$$H'_{x,y}(t) = 1 + (x-y)^2 \int_0^\infty u e^{-(x+y+2t)u} \,\mathrm{d}\, u, \tag{3.7}$$

and so, by integrating with respect to $t \in (0, s)$ on both sides of (3.7), the formula (3.1) follows.

Letting $s \to \infty$ on both sides of (3.1) and using the limit $\lim_{s\to\infty}[H_{x,y}(s) - s] = A(x, y)$ generate the formula (3.2).

Taking $x \to 0^+$ in (3.1) produces (3.3). Theorem 3.1 is thus proved. □

Remark 3.1. By [7, pp. 161–162, Theorem 3] or [25, p. 45, Proposition 5.17], it can be derived that the reciprocal of the harmonic mean $H_{x,y}(t)$, that is, the function $\frac{1}{A(1/(x+t),1/(y+t))}$, is logarithmically completely monotonic.

This logarithmically complete monotonicity can also be proved by considering

$$[\ln H_{x,y}(t)]' = \frac{x^2 + y^2 + 2(x+y)t + 2t^2}{(x+t)(y+t)(x+y+2t)} = \frac{1}{2}\left(\frac{1}{x+t} + \frac{1}{y+t}\right)\left[1 + \frac{(x-y)^2}{(x+y+2t)^2}\right]$$

and that the product and sum of finitely many completely monotonic functions are also completely monotonic functions.

Moreover, from (3.4), it follows readily that $H_{x,y}(t) - t$ is an increasing function in $t \in (-\min\{x, y\}, \infty)$ for $x, y > 0$ with $x \ne y$.

4. The geometric mean is a Bernstein function

Our results on the geometric mean $G_{x,y}(t)$ can be summarized as two theorems.

Theorem 4.1. *Let $x, y > 0$ with $x \ne y$. Then the geometric mean $G_{x,y}(t)$ defined by (1.16) is a Bernstein function of t on $(-\min\{x, y\}, \infty)$.*

We supply three proofs of Theorems 4.1.

First proof. By a direct differentiation, we have

$$G'_{x,y}(t) = \sqrt{\frac{x+t}{y+t}}\, \frac{x+y+2t}{2(x+t)}.$$

Taking the logarithm on both sides of the above equality creates

$$\ln G'_{x,y}(t) = \frac{1}{2} \ln \frac{x+t}{y+t} + \ln \frac{x+y+2t}{2(x+t)}. \tag{4.1}$$

In [1, p. 230, 5.1.32], it was collected that for $a > 0$ and $b > 0$,

$$\ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} \,\mathrm{d}\, u. \tag{4.2}$$

Using this formula in (4.1) leads to

$$\ln G'_{x,y}(t) = \int_0^\infty \frac{e^{-(x+t)v} + e^{-(y+t)v} - 2e^{-v[(x+t)+(y+t)]/2}}{2v} \,\mathrm{d}\, v.$$

Since the function e^{-t} is convex on $\mathbb{R}$, we have

$$e^{-(x+t)v} + e^{-(y+t)v} - 2e^{-v[(x+t)+(y+t)]/2} \ge 0.$$

Therefore, we have

$$[\ln G'_{x,y}(t)]^{(k)} = \frac{(-1)^k}{2}\int_0^\infty \bigl\{e^{-(x+t)v} + e^{-(y+t)v} - 2e^{-v[(x+t)+(y+t)]/2}\bigr\}v^{k-1}\,\mathrm{d}\,v.$$

This means that the derivative $G'_{x,y}(t)$ is logarithmically completely monotonic, and so it is also completely monotonic. As a result, the geometric mean $G_{x,y}(t)$ is a Bernstein function. □

Second proof. It is clear that the geometric mean $G_{x,y}(t)$ satisfies

$$G'_{x,y}(t) = \frac{1}{2}\left(\sqrt{\frac{x+t}{y+t}} + \sqrt{\frac{y+t}{x+t}}\right) = \frac{1}{2}\left(\sqrt{u} + \frac{1}{\sqrt{u}}\right) \triangleq f(u) \tag{4.3}$$

and

$$[\ln G_{x,y}(t)]' = \frac{1}{2}\left(\frac{1}{x+t} + \frac{1}{y+t}\right), \tag{4.4}$$

where

$$u \triangleq u_{x,y}(t) = \frac{x+t}{y+t} = 1 + \frac{x-y}{y+t}. \tag{4.5}$$

If $0 < x < y$, then $0 < u_{x,y}(t) < 1$ for $t \in (-x,\infty)$ and $u'_{x,y}(t) = \frac{y-x}{(y+t)^2}$ is completely monotonic in $t \in (-x,\infty)$. On the other hand, the function $f(u)$ is positive and

$$\begin{aligned} f^{(i)}(u) &= \frac{1}{2}\left[(-1)^{i-1}\frac{(2i-3)!!}{2^i}u^{-(2i-1)/2} + (-1)^i\frac{(2i-1)!!}{2^i}u^{-(2i+1)/2}\right] \\ &= \frac{(-1)^i(2i-3)!!}{2^{i+1}}\frac{1}{u^{(2i-1)/2}}\left(\frac{2i-1}{u} - 1\right) \end{aligned}$$

for $i \in \mathbb{N}$, which implies that the function $f(u)$ is completely monotonic on $(0,1)$; A ready modification of a conclusion in [5, p. 83] yields the following conclusion: If g and h' are completely monotonic functions such that $g(h(x))$ is defined on an interval I, then $x \mapsto g(h(x))$ is also completely monotonic on I; So, when $y > x > 0$, the derivative $G'_{x,y}(t)$ is completely monotonic and the geometric mean $G_{x,y}(t)$ is a Bernstein function. Consequently, considering the symmetric property $G_{x,y}(t) = G_{y,x}(t)$, it is easily obtained that the geometric mean $G_{x,y}(t)$ for $t \in (-\min\{x,y\},\infty)$ with $x \ne y$ is a Bernstein function. □

Remark 4.1. From the equality in (4.3), it is easy to derive that the function $G_{x,y}(t) - t$ is increasing in $t \in (-\min\{x,y\},\infty)$ for $x, y > 0$ with $x \ne y$.

From (4.4), it is immediate to deduce that the reciprocal of the geometric mean $G_{x,y}(t)$ is a logarithmically completely monotonic function of $t \in (-\min\{x,y\},\infty)$ for $x, y > 0$ with $x \ne y$.

Third proof. By (4.3) and (4.5), it follows that

$$G'_{x,y}(t) = \frac{1}{2}\left[h\left(\frac{y+t}{x-y}\right) + \frac{1}{h\bigl(\frac{y+t}{x-y}\bigr)}\right] = \frac{1}{2}H\left(\frac{y+t}{x-y}\right) \tag{4.6}$$

and

$$[G'_{x,y}(t)]^{(i)} = \frac{1}{2(x-y)^i}H^{(i)}\left(\frac{y+t}{x-y}\right)$$

for $i \in \{0\} \cup \mathbb{N}$. By the formula (2.5) in Lemma 2.1, we have

$$
\begin{aligned}
[G'_{x,y}(t)]^{(i)} = \frac{(-1)^i}{2^{i+1}\bigl(\frac{y+t}{x-y}\bigr)^{i+1}(x+t)^i h\bigl(\frac{y+t}{x-y}\bigr)} \\
\times \sum_{k=0}^{i-1} \frac{(i-1)!(i+1)!(2i-2k-1)!!}{(i-k-1)!(i-k+1)!k!} 2^k \left(\frac{y+t}{x-y}\right)^k,
\end{aligned}
$$

which means that, when $x > y$, the derivative $G'_{x,y}(t)$ is completely monotonic. Since $G_{x,y}(t) = G_{y,x}(t)$, when $x < y$, the derivative $G'_{y,x}(t)$ is also completely monotonic. This implies that the geometric mean $G_{x,y}(t)$ is a Bernstein function of $t \in (-\min\{x,y\},\infty)$. $\square$

Theorem 4.2. *For $x > y > 0$ and $z \in \mathbb{C} \setminus (-\infty,-y]$, the geometric mean $G_{x,y}(z)$ has the integral representation*

$$
G_{x,y}(z) = G(x,y) + z + \frac{x-y}{2\pi}\int_0^\infty \frac{\rho((x-y)s)}{s} e^{-ys}\bigl(1-e^{-sz}\bigr)\,\mathrm{d}\,s, \tag{4.7}
$$

where the function ρ is defined by (2.36)*. Consequently, the geometric mean $G_{x,y}(t)$ is a Bernstein function of t on $(-\min\{x,y\},\infty)$.*

Proof. For $x > y > 0$ and $z \in \mathbb{C} \setminus (-\infty,-y]$, making use of

$$
G'_{x,y}(z) = \frac{1}{2}\left[h\left(\frac{y+z}{x-y}\right) + \frac{1}{h\bigl(\frac{y+z}{x-y}\bigr)}\right] = \frac{1}{2}H\left(\frac{y+z}{x-y}\right)
$$

and (2.35) gives

$$
G'_{x,y}(z) = 1 + \frac{1}{2\pi}\int_0^\infty \rho(s)\exp\left(-\frac{y+z}{x-y}s\right)\mathrm{d}\,s.
$$

Integrating with respect to z from 0 to w on both sides of the above equation and interchanging the order of integrals yield

$$
\begin{aligned}
G_{x,y}(w) - G_{x,y}(0) &= w + \frac{x-y}{2\pi}\int_0^\infty \frac{\rho(s)}{s}\exp\left(-\frac{ys}{x-y}\right)\left[1-\exp\left(-\frac{sw}{x-y}\right)\right]\mathrm{d}\,s \\
&= w + \frac{x-y}{2\pi}\int_0^\infty \frac{\rho((x-y)s)}{s} e^{-ys}\bigl(1-e^{-ws}\bigr)\,\mathrm{d}\,s.
\end{aligned}
$$

Since $G_{x,y}(0) = G(x,y)$, the integral representation (4.7) is readily deduced.

By the characterization expressed by (1.7) and the integral representation (4.7) applied to $z = t \in (-\min\{x,y\},\infty)$, it is immediate to see that the geometric mean $G_{x,y}(t)$ is a Bernstein function of t on $(-\min\{x,y\},\infty)$. $\square$

Remark 4.2. Taking $z \to \infty$ in (4.7) and using $\lim_{z\to\infty}[G_{x,y}(z) - z] = A(x,y)$ yield

$$
A(x,y) = G(x,y) + \frac{x-y}{2\pi}\int_0^\infty \frac{\rho((x-y)s)}{s} e^{-ys}\,\mathrm{d}\,s \ge G(x,y). \tag{4.8}
$$

The equality in (4.8) is valid if and only if $x = y$. This gives a new proof of the fundamental and well known AG mean inequality.

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 9th printing, Washington, 1970.

[2] R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.

[3] C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439.

[4] C. Berg, *Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity*, In: *Positive Definite Functions. From Schoenberg to Space-Time Challenges*, S. Mateu and E. Porcu (eds.), Dept. Math., Univ. Jaume I, Castellón de la Plana, Spain, 2008, 24 pages.

[5] S. Bochner, *Harmonic Analysis and the Theory of Probability*, University of California Press, Berkeley-Los Angeles, 1960.

[6] P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and its Applications, Volume 560, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003.

[7] C.-P. Chen, F. Qi, and H. M. Srivastava, *Some properties of functions related to the gamma and psi functions*, Integral Transforms Spec. Funct. **21** (2010), no. 2, 153–164; Available online at `http://dx.doi.org/10.1080/10652460903064216`.

[8] T. W. Gamelin, *Complex Analysis*, Undergraduate Texts in Mathematics, Springer, New York-Berlin-Heidelberg, 2001.

[9] B.-N. Guo and F. Qi, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218** (2012), no. 19, 9890–9897; Available online at `http://dx.doi.org/10.1016/j.amc.2012.03.075`.

[10] B.-N. Guo and F. Qi, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.

[11] B.-N. Guo and F. Qi, *A simple proof of logarithmic convexity of extended mean values*, Numer. Algorithms **52** (2009), no. 1, 89–92; Available online at `http://dx.doi.org/10.1007/s11075-008-9259-7`.

[12] B.-N. Guo and F. Qi, *The function* $(b^x - a^x)/x$*: Logarithmic convexity and applications to extended mean values*, Filomat **25** (2011), no. 4, 63–73; Available online at `http://dx.doi.org/10.2298/FIL1104063G`.

[13] B.-N. Guo and F. Qi, *The function* $(b^x - a^x)/x$*: Ratio's properties*, Available online at `http://arxiv.org/abs/0904.1115`.

[14] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 2, John Willey & Sons, 1977.

[15] G. A. Kalugin, D. J. Jeffrey, and R. M. Corless, *Bernstein, Pick, Poisson and related integral expressions for Lambert W*, Integral Transforms Spec. Funct. **23** (2012), no. 11, 817–829; Available online at `http://dx.doi.org/10.1080/10652469.2011.640327`.

[16] G. A. Kalugin, D. J. Jeffrey, R. M. Corless, and P. B. Borwein, *Stieltjes and other integral representations for functions of Lambert W*, Integral Transforms Spec. Funct. **23** (2012), no. 8, 581–593; Available online at `http://dx.doi.org/10.1080/10652469.2011.613830`.

[17] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[18] F. Qi, *A new lower bound in the second Kershaw's double inequality*, J. Comput. Appl. Math. **214** (2008), no. 2, 610–616; Availbale online at `http://dx.doi.org/10.1016/j.cam.2007.03.016`.

[19] F. Qi, *The extended mean values: Definition, properties, monotonicities, comparison, convexities, generalizations, and applications*, Cubo Mat. Educ. **5** (2003), no. 3, 63–90.

[20] F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, J. Math. Anal. Appl. **296** (2004), no. 2, 603–607; Available online at `http://dx.doi.org/10.1016/j.jmaa.2004.04.026`.

[21] F. Qi and S.-X. Chen, *Complete monotonicity of the logarithmic mean*, Math. Inequal. Appl. **10** (2007), no. 4, 799–804; Available online at `http://dx.doi.org/10.7153/mia-10-73`.

[22] F. Qi and B.-N. Guo, *Complete monotonicities of functions involving the gamma and digamma functions*, RGMIA Res. Rep. Coll. **7** (2004), no. 1, Art. 8, 63–72; Available online at `http://rgmia.org/v7n1.php`.

[23] F. Qi, S. Guo, and S.-X. Chen, *A new upper bound in the second Kershaw's double inequality and its generalizations*, J. Comput. Appl. Math. **220** (2008), no. 1-2, 111–118; Available online at http://dx.doi.org/10.1016/j.cam.2007.07.037.
[24] F. Qi, C.-F. Wei, and B.-N. Guo, *Complete monotonicity of a function involving the ratio of gamma functions and applications*, Banach J. Math. Anal. **6** (2012), no. 1, 35–44.
[25] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, de Gruyter Studies in Mathematics **37**, De Gruyter, Berlin, Germany, 2010.
[26] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.
[27] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.

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